Pre-class Warm-up!!!

Which of the following systems of equations is equivalent to the 2nd order equation
$x'' - 3x' + 2x = 0$?

✓ a. $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

b. $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

c. $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

d. None of the above.

Another one: what is the Wronskian of
$\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$?

a. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} e^{2t}$

b. $-2e^{-3t}$

✓ c. $-2e^{2t}$

d. $\begin{bmatrix} 2 \\ 0 \end{bmatrix} (e^{3t} + e^{-t})$

Section 7.3: the eigenvalue method for linear systems

We learn:

- how to solve homogeneous first order linear systems with constant coefficients using eigenvalues and eigenvectors

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Solve $x_1' = x_1 + 2x_2$
$x_2' = 2x_1 + x_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution. Find the e-values and e-vectors of $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Char. poly. $\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$

$= (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$

The e-vector for $\lambda = 3$. $\mathrm{Null}\left(\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix}\right)$

$= \mathrm{Null} \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$ has basis $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The e-vector for $\lambda = -1$: $\mathrm{Null} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ has basis $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}.$$

Why does this work?

If v is an e-vector with e-value λ,

then $Av = \lambda v$.

Thus $\underline{x} = v e^{\lambda t}$ has

$\underline{x}' = \lambda v e^{\lambda t} = A v e^{\lambda t} = A \underline{x}$

Another one:

Knowing that P^-1 A P = D where

$$A = \begin{bmatrix} -5 & -14 \\ 3 & 8 \end{bmatrix} \quad P = \begin{bmatrix} 7 & 2 \\ -3 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

solve the system

$$x_1' = -5x_1 - 14x_2$$
$$x_2' = 3x_1 + 8x_2$$

Which of the following are solutions?

a. $\begin{bmatrix} 7e^t \\ -3e^{2t} \end{bmatrix}$ and $\begin{bmatrix} 2e^{2t} \\ -e^t \end{bmatrix}$

b. $\begin{bmatrix} -5 \\ 3 \end{bmatrix} e^t$ and $\begin{bmatrix} -14 \\ 8 \end{bmatrix} e^{2t}$

✓c. $\begin{bmatrix} 7 \\ -3 \end{bmatrix} e^t$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{2t}$

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Find the general solution to

$$x_1' = 5x_1 - 9x_2$$
$$x_2' = 2x_1 - x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution. Char. poly. $\det\begin{bmatrix} 5-\lambda & -9 \\ 2 & -1-\lambda \end{bmatrix}$

$$= \lambda^2 - 4\lambda - 5 + 18 = \lambda^2 - 4\lambda + 13$$

$$\lambda = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$$

The e-vector for $2+3i$:

$$\text{Null}\begin{bmatrix} 3-3i & -9 \\ 2 & -3-3i \end{bmatrix}$$

Note $(3-3i)(3+3i) = 9+9 = 18$

$$② \to \frac{3+3i}{9}① + ② \qquad \begin{bmatrix} 3-3i & -9 \\ 0 & 0 \end{bmatrix}$$

Nullspace has basis $\begin{bmatrix} 3+3i \\ 2 \end{bmatrix}$

The other e-vector for $\lambda = 2-3i$ is the complex conjugate of the first e-vector:

$$\begin{bmatrix} 3-3i \\ 2 \end{bmatrix}$$

General solution: $A\begin{bmatrix} 3+3i \\ 2 \end{bmatrix} e^{(2+3i)t} + B\begin{bmatrix} 3-3i \\ 2 \end{bmatrix} e^{(2-3i)t}$

$$\begin{bmatrix} 3+3i \\ 2 \end{bmatrix} e^{2t}(\cos 3t + i\sin 3t) = e^{2t}\begin{bmatrix} 3\cos 3t - 3\sin 3t \\ 2\cos 3t \end{bmatrix} + ie^{2t}\begin{bmatrix} 3\sin 3t + 3\cos 3t \\ 2\sin 3t \end{bmatrix}$$

is a solution. So is its complex conjugate

$$\left[(\text{This solution}) + \overline{(\text{This solution})}\right] / 2 = e^{2t}\begin{bmatrix} 3\cos 3t - 3\sin 3t \\ 2\cos 3t \end{bmatrix}$$

$$(\text{This soln}) - \overline{(\text{this soln})} / 2i = e^{2t}\begin{bmatrix} 3\sin 3t + 3\cos 3t \\ 2\sin 3t \end{bmatrix}$$

General solution $e^{2t}\left(c_1\begin{bmatrix} 3\cos 3t - 3\sin 3t \\ 2\cos 3t \end{bmatrix} + c_2\begin{bmatrix} 3\sin 3t + 3\cos 3t \\ 2\sin 3t \end{bmatrix}\right)$

Here it is again, more neatly:

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Find the general solution to
$$x_1' = 5x_1 - 9x_2$$
$$x_2' = 2x_1 - x_2$$

Solution: find the eigenvalues.
The characteristic polynomial is

$$\det\begin{bmatrix} 5-\lambda & -9 \\ 2 & -1-\lambda \end{bmatrix} = \lambda^2 - 4\lambda - 5 + 18 = \lambda^2 - 4\lambda + 13$$

$$\lambda = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$$

Eigenvectors: for $\lambda = 2+3i$ we want a basis for the nullspace of

$$\begin{bmatrix} 3-3i & -9 \\ 2 & -3-3i \end{bmatrix}$$ Note $(3-3i)(3+3i) = 18$

$$② \to ② - \frac{(3+3i)}{9} ① \quad \begin{bmatrix} 3-3i & -9 \\ 0 & 0 \end{bmatrix}$$

The nullspace has basis $\begin{bmatrix} 3(1+i) \\ 2 \end{bmatrix}$

The complex conjugate $\lambda = 2-3i$ has eigenvector $\begin{bmatrix} 3(1-i) \\ 2 \end{bmatrix}$

We get solutions $\begin{bmatrix} 3+3i \\ 2 \end{bmatrix} e^{(2+3i)t}$

$$= e^{2t}(\cos 3t + i\sin 3t)\begin{bmatrix} 3+3i \\ 2 \end{bmatrix}$$

$$= e^{2t}\left(\begin{bmatrix} 3\cos 3t - 3\sin 3t \\ 2\cos 3t \end{bmatrix} + i\begin{bmatrix} 3\cos 3t + 3\sin 3t \\ 2\sin 3t \end{bmatrix}\right)$$

and the complex conjugate
- - same - - - $-i$ - same -

Add and $\div 2$ or subtract and $\div 2i$ to get solutions

$$e^{2t}\begin{bmatrix} 3\cos 3t - 3\sin 3t \\ 2\cos 3t \end{bmatrix} \text{ and } e^{2t}\begin{bmatrix} 3\cos 3t + 3\sin 3t \\ 2\sin 3t \end{bmatrix}$$

Question:
Consider a system $x' = Ax$ where A has an eigenvector $\begin{bmatrix} 1 \\ 2-i \end{bmatrix}$ with eigenvalue $\lambda = 1 + 3i$

1. Which of the following must also be an eigenvalue?

a. $3 + i$
b. $1 - 3i$
c. $-1 + 3i$

2. Which of the following must also be an eigenvector?

a. $\begin{bmatrix} 1 \\ 1+3i \end{bmatrix}$

b. $\begin{bmatrix} 1 \\ 2+i \end{bmatrix}$

c. $\begin{bmatrix} 2-i \\ 1 \end{bmatrix}$

d. $\begin{bmatrix} 2-i \\ 5 \end{bmatrix}$

3. Which of the following are solutions?

a. $e^t \begin{bmatrix} \cos 3t + i \sin 3t \\ 2\cos 3t + \sin 3t + i(2\sin 3t - \cos 3t) \end{bmatrix}$

b. $e^t \begin{bmatrix} \cos 3t + i \sin 3t \\ 2\cos 3t + i \sin 3t \end{bmatrix}$

4. Which of the following are solutions?

a. $e^t \begin{bmatrix} \cos 3t \\ 2\sin 3t \end{bmatrix}$

b. $e^t \begin{bmatrix} \cos 3t \\ 2\cos 3t + \sin 3t \end{bmatrix}$